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BLOCK CODES FOR BERNOULLI SHIFTS

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ABSTRACT
A general method of constructing block codes between Bernoulli shifts is introduced. This method generalizes an example of Boyle and Tuncel.

1. Introduction

Let A and B be finite sets. Consider the spaces $X = A^{\mathbf{Z}}$ and $Y = B^{\mathbf{Z}}$. Let S be the left shift on X and T the left shift on Y. A k-block code is a map

$$\phi : A^k \to B.$$

It gives rise to an equivariant continuous map from X to Y, namely

$$(\phi x)_i = \phi(x_i, \dots, x_{i+k-1}), \qquad \text{where } x = \{x_i\} i \in z.$$

It is known that every equivariant continuous map is a block map composed with a power of the shift S.

If p is a probability measure on A, that is p is a probability vector, and if $\mu = p^z$ is the product measure on $A^{\mathbf{Z}}$, then S acting on the measure space is a *Bernoulli shift* (B.S.) denoted by $B(p)$. Let q be a probability vector for B. $B(q)$ is a continuous factor of $B(p)$ if there exists a k-block code ϕ such that $p^{\mathbf{Z}}$ is carried onto $q^{\mathbf{Z}}$ by the homomorphism ϕ. We shall say that $B(q)$ is a trivial factor of $B(p)$ if there is a map,

$$f : A \to B \qquad \text{such that } q(b) = \sum_{f(a)=b} p(a), \quad \forall b \in B,$$

i.e., the vector q is a clustering of p. Obviously in such a case there exists a 1-block map from $B(p)$ onto $B(q)$.

This paper is a contribution to the study of continuous Bernoulli factors which was initiated in [3].

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If $B(q)$ is continouos factor of $B(p)$ it is evident that $|A| \geqq |B|$ since *topological entropy* decreases under taking factors. Suppose $|A| = |B|$. Considering metrical entropy we know that $h(p) \geqq h(q)$. The case $h(p) = h(q)$ was settled in [3] and it was there proved that q is a rearrangement of p, and therefore there are no non-trivial factors in such a case. The case $|A| = |B|$ and $h(p) > h(q)$ is impossible since by [2] a continuous equivariant map which preserves the topological entropy is finite to one. But a finite to one map preserves the metrical entropy.

In [3] it was proved that any continuous factor of the n-shift $B(1/n, \dots, 1/n)$ is trivial. It was conjectured in [3] that there are no non-trivial continuous factors. However, in [1] an example was constructed of a non-trivial continuous factor (see §3).

In this paper a method of constructing factors for B.S.s is presented. This "tree method" gives us examples of a 2-state B.S. as a continuous factor of a 4-state B.S., also a 3-state B.S. as a factor of a 4-state B.S. We also indicate how to get k-block maps between B.S.s which do not admit k-1-block maps.

2. The tree method

A k-tree T is a set of k-tuples $(i_1, \dots, i_k)$ of natural numbers (called *branches*) subject to the following restrictions. There is a natural number $n(\emptyset)$ and $1 \leqq i_1 \leqq n(\emptyset)$ and for each $(i_1, \dots, i_j)$, $1 \leqq j < k$ which is an initial segment of a branch there exists a natural number $n(i_1, \dots, i_j)$ and $1 \leqq i_{i+1} \leqq n$ $(i_1, \dots, i_j)$. Observe that for each $1 \leqq j < k$ there is a j-tree associated with T, namely the j-tuples $(i_1, \dots, i_j)$ which are the initial segments of branches of T. This tree be denoted by $T(j)$.

A *probability tree* is a tree together with a set of probability vectors $\{p(i_1, \dots, i_j)\}$, $0 \leqq j < k$ such that the number of components of $p(i_1, \dots, i_j)$ is equal to $n(i_1, \dots, i_1)$. Given a probability tree T we associate with it the probability vector of T, $p(T)$, in the following way. $p(T)$ is labelled by the branches of T and

$$p(i_1, \dots, i_k)(T) = \prod_{j=0}^{k-1} P_{i_{j+1}}(i_1, \dots, i_j).$$

Given two probability vectors p defined on A, and q defined on B, we say that q is a *clustering* of p, denoted $q \prec p$, if there exists a map $f : A \to B$ such that

$$q_b = \sum_{f(a)=b} p_a, \qquad \forall b \in B.$$

The product of p and q, denoted by $p \cdot q$, is the probability vector on $A \times B$, $\{p_a \cdot p_b\}$. We are now ready to state the main result of this paper.

THEOREM. *Let T be a probability k-tree and $p = p(T)$. Let q be another probability vector such that for all $(\alpha_0, \ldots, \alpha_{k-1}) \in \Pi_{j=0}^{k-1} T(j)$ (where Π is the cartesian product)*

$$(*) \qquad q < \prod_{j=0}^{k-1} p(\alpha_j).$$

Then, there exists a k-block map from $B(p)$ onto $B(q)$.

PROOF. Let us take the state space A to be the branches of T. By condition $(*)$ there are maps

$$f_{(\alpha_0, \ldots, \alpha_{k-1})} : \prod_{j=1}^{k-1} T(j) \to B, \qquad \alpha_j \in T(j).$$

(Notice that here $\{\alpha_j\}$ are not necessarily sub-branches of the same branch in T.)

We define the k-block map ϕ as follows:

Let $x_n = (i_1, \ldots, i_k)$; define the random variables

$$Z_n^{(j)} = (i_1, \ldots, i_j), \quad 1 \leqq j \leqq k \quad \text{and} \quad Z_n^{(0)} = \emptyset.$$

Put

$$\phi(x_1, \ldots, x_k) = f_{(Z_k^{(0)}, \ldots, Z_1^{(k-1)})}(Z_k^{(1)}, \ldots, Z_1^{(k)}).$$

We shall prove by induction on n the following two claims:

(1) $y_1, \ldots, y_n$ are independent

(2) $(y_1, \ldots, y_n)$ is independent of $(Z_1^{(k-1)}, \ldots, Z_{k-1}^{(1)})$.

For $n = 1$, (2) follows from the clustering assumption $(*)$. Assume that (1) and (2) hold for some $n \geqq 1$. Consider $y_1, \ldots, y_{n+1}$. By stationarity $y_2, \ldots, y_{n+1}$ are independent and $(y_2, \ldots, y_{n+1})$ is independent of $(Z_2^{k-1}, \ldots, Z_k^{(1)})$. Since $(Z_2^{(k-1)}, \ldots, Z_k^{(1)}, y_2, \ldots, y_{n+1})$ is independent of $(x_1, Z_1^{(k-1)})$ it follows that $(y_2, \ldots, y_{n+1})$ is independent of $(X_1, Z_1^{(k-1)} Z_2^{(k-1)}, \ldots, Z_k^{(1)})$. But Y_1 is $(X_1 Z_1^{(k-1)}, Z_2^{(k-1)}, \ldots, Z_k^{(1)})$-measurable and therefore $Y_1, \ldots, Y_{n+1}$ are independent. So claim (1) holds for $n+1$. Also $(Z_1^{(k-1)}, \ldots, Z_{k-1}^{(1)} Y_1)$ is $(X_1 Z_2^{(k+1)}, \ldots, Z_k^{(1)})$-measurable and therefore independent of $y_2, \ldots, y_{n+1}$. Now, y_1 is independent of $(Z_1^{(k-1)}, Z_2^{(k-1)}, \ldots, Z_{k-1}^{(1)})$ (Claim (2) for $n = 1$) and so $y_1, \ldots, y_{n+1}$ is independent of $(Z_1^{(k+1)}, \ldots, Z_{k-1}^{(1)})$ and the proof of the theorem is completed.

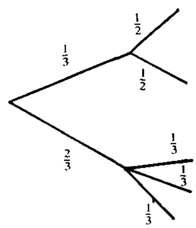
3. Examples

We begin by some 2-block codes which produce non-trivial factors.

(1) The Boyle–Tuncel example ([1]).

Let $p = (\frac{1}{6}, \frac{1}{6}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9})$ and $q = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

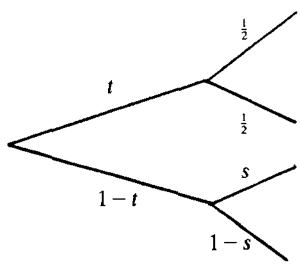
Represent p by the following tree:



Since $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) < (\frac{1}{2}, \frac{1}{2}) \cdot (\frac{1}{2}, \frac{2}{3})$ and trivially $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) < (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \cdot (\frac{1}{2}, \frac{1}{2})$, there exists a 2-block map from $B(p)$ onto $B(q)$.

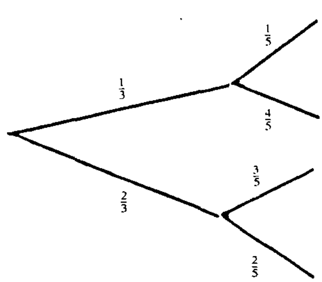
(2) 2-state B.S. as a factor of a family of 4-state B.S.s.

Let $\frac{1}{2} < t < 1$ and $s = 1/2t$. Consider the following tree:



Since $(\frac{1}{2}, \frac{1}{2}) \preccurlyeq (s, 1-s) \cdot (t, 1-t)$ there is a 2-block from $B(\frac{1}{2}t, \frac{1}{2}t, (1-t)s, (1-t)(1-s))$ onto $B(\frac{1}{2}, \frac{1}{2})$ for each $\frac{1}{2} < t < 1$. If we put $t = \frac{2}{3}$ we get $B(\frac{1}{2}, \frac{1}{2})$ as a non-trivial factor of $B(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{12})$. Actually, the same code will produce $B(\frac{1}{2}, \frac{1}{2})$ as a non-trivial factor of all the $B(p)$ obtained from the above tree.

(3) As an example of achieving minimal difference between the number of states of $B(p)$ and its non-trivial factor $B(q)$ consider the tree:

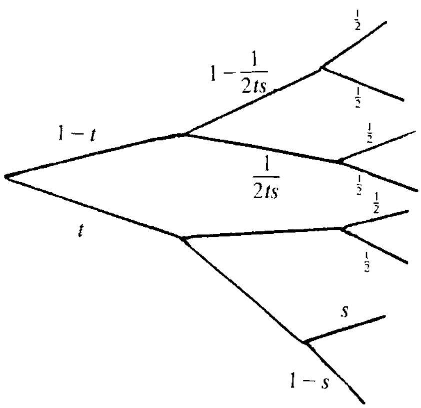


Since $(\frac{8}{15}, \frac{4}{15}, \frac{1}{5}) < (\frac{1}{5}, \frac{4}{5}) \cdot (\frac{1}{3}, \frac{2}{3})$; $(\frac{2}{5}, \frac{3}{5}) \cdot (\frac{1}{3}, \frac{2}{3})$ there is a 2-block map from $B(\frac{2}{5}, \frac{4}{15}, \frac{4}{15}, \frac{1}{15})$ onto $B(\frac{8}{15}, \frac{4}{15}, \frac{1}{5})$.

This code can also produce uncountable pairs of B.S.s where one is the non-trivial factor of the other.

Now we consider some higher block maps.

(4) Let $\frac{1}{2} < t < s < 1$. Consider the following tree:



It is not hard to check that $q = (\frac{1}{2}, \frac{1}{2})$ meets the conditions of the theorem for such a tree. If $p = p(T)$ for that tree T there is a 3-block map from $B(p)$ onto $B(\frac{1}{2}, \frac{1}{2})$.

Observe that in the probability vector p^2 which is a function of (s, t) there is no subset of components the sum of which identically (in (s, t)) equals $\frac{1}{2}$. Therefore there can be at most a finite number of values (s, t) such that there will be a 2-block map from $B(p)$ onto $B(\frac{1}{2}, \frac{1}{2})$.

(5) We now generalize example (4) in the following way.

Let $\frac{1}{2} < S_0 < S_2 \cdots < S_{k-1} = 1$ be given. Let T be the following k-tree: $T = (i, \ldots, i_k)$, $i_j \in \{1, 2\}$. Let $i \leqq j \leqq k - 1$,

$$p(\underbrace{2, 2, \ldots, 2}_{j \text{ times}}) = \left(\frac{s_{j-1}}{s_j}, 1 - \frac{s_{j-1}}{s_j} \right) \quad \text{and} \quad p(i, \ldots, i_j) = \left(\frac{1}{2s_j}, 1 - \frac{1}{2s_j} \right)$$

if $(i, \ldots, i_j) = (2, 2, \ldots, 2)$ and $p(0) = (1/2s_0, 1 - 1/2s_0)$. Again $q = (\frac{1}{2}, \frac{1}{2})$ satisfies the conditions of the theorem for that tree and there is a k-block map from $B(p(T))$ onto $B(\frac{1}{2}, \frac{1}{2})$. $p^{k-1}(T)$ as a function of $(s_0, \ldots, s_{k-2})$ will not have a subset the sum of which will be identically $\frac{1}{2}$ and therefore there will be at most finite number of $k - 1$ tuplets $(s_0, \ldots, s_{k-2})$ for which there will be a $k - 1$-block code from $P(T)$ onto $B(\frac{1}{2}, \frac{1}{2})$.

4. Some open problems

(1) Given $B(q)$ a continuous factor of $B(p)$, does there exist a probability tree T such that $p = p(T)$ and q satisfies the conditions of the theorem?

A positive answer to problem (1) will give an effective way to find all the continuous factors of a given $B(p)$.

(2) Is there a non-trivial (2-state) B.S. factor of a 3-state B.S.?

Again, if (1) admits a positive answer, then it is easy to see that there will be no non-trivial factors, because a 3-state B.S. will admit only 2-trees and q would have to be a clustering of p.

(3) Is there a $2k$ ($k \geqq 1$) B.S. which is a non-trivial continuous factor of a $2k+1$ state B.S.?

Using a method similar to example (3) one can construct a non-trivial $2k-1$-state B.S. as continuous factors of a $2k$-state B.S. for all $k \geqq 2$.

References

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